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Co-H-MAPS TO SPHERES

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ABSTRACT
Criteria are given which characterize Co-H and Co-A maps from arbitrary double suspensions to odd dimensional spheres in terms of the maps in the EHP sequence.

Introduction

Let $f: \Sigma^2 X \to S^{2n+1}$ be a map. We assume that spaces are p-local, with p an odd prime, but otherwise X is arbitrary of finite type. Our first result is

THEOREM 1. *f is a co-H-map if and only if $H \circ f^*$ is null-homotopic.*

Here $H: \Omega S^{2n+1} \to \Omega S^{2np+1}$ is any Hopf invariant and f^* denotes the single adjoint of f.

Our main topic is the notion of a co-A-map. The definition is deferred, but roughly, co-A-maps are co-H-maps with a suitable homotopy making the induced comultiplication on the mapping cone homotopy associative. Our aim is to establish a result like Theorem 1 for co-A-maps. We wish to characterize the co-A-maps to spheres in terms of the standard invariants of homotopy theory.

In [H], it was shown that the maps constructed by Selick [S_1] fit into the following homotopy commutative diagram,

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$$
\begin{array}{ccccccc}
\Omega^3 S^{2np+1} & \xrightarrow{\Omega\partial} & \Omega J_{p-1}S^{2n} & \xrightarrow{\Omega i} & \Omega^2 S^{2n+1} & \xrightarrow{\Omega H} & \Omega^2 S^{2np+1} \\
 & \searrow^{\Omega^3 p} & \downarrow{\scriptstyle \Omega E^2 \circ T} & & \downarrow{\scriptstyle S} & \nearrow & \\
 & & \Omega^3 S^{2np+1} & \longrightarrow & \Omega^2 S^{2np+1}\{p\} & &
\end{array}
$$

provided T is chosen to be an H-map [G$_1$]. Here, the top row is from Toda's fibration and the bottom row is from the fibration for the degree p map of S^{2np+1}. Our main result is

THEOREM 2. *$f\colon \Sigma^2 X \to S^{2n+1}$ is a co-A-map if and only if $S \circ f^{**}$ is null-homotopic where f^{**} is the double adjoint.*

A reformulation of Theorem 2 which is useful for calculations is 'f is a co-A-map if and only if $\Omega E^2 \circ T \circ \bar{f}$ is divisible by p' where

$$\bar{f}\colon X \to \Omega J_{p-1} S^{2n}$$

is any lift of f^{**}. This contrasts with a result from [S$_2$] that f is homotopic to a double suspension if and only if there exists some lift $\bar{f}$ such that $T \circ \bar{f}$ is divisible by p. Thus qualitatively, the difference between co-A-maps and double suspensions is a double suspension. In the case of p-local two-cell complexes of the form

$$Y = S^{2n+1} \cup e^r$$

this difference is present on the mapping cone. In [BH] we show that Y admits the structure of a cogroup if and only if the attaching map is a co-A-map, and examples not homotopic to any suspension are given.

From the main result of [CMN], we know that multiplication by p, in the form $f \circ pI$, is always homotopic to a double suspension. Our Theorem 2 yields the weaker result that $f \circ pI$ is always a co-A-map.

As in other parts of homotopy theory, the three-sphere is special for this theory.

THEOREM 3. *$f\colon \Sigma^2 X \to S^3$ is a co-A-map if and only if there is $g\colon X \to S^1$ such that $f \simeq \Sigma^2 g$. In particular $\Pi_* S^3$ has no essential co-A-maps of odd order.*

A somewhat technical, but sharper form of Theorem 3 is the following result. Let α_1 generate $\Pi_{2p} S^3$.

THEOREM 4. *Let $f\colon X \to S^{2p-1}$. Then $\Sigma^2 f$ is divisible by p in $[\Sigma^2 X, S^{2p+1}]$ if and only if $\alpha_1 \circ \Sigma f$ is null-homotopic.*

The paper is organized in three sections. The proof of Theorem 1 is given in the first section. This result is essentially already in the literature [BH], [G_1].

The notion of the co-A-deviation is developed in the second section. There are two parts to this discussion. The first concerns geometric properties while the second is algebraic. In the third section, the problem is reduced to the study of a universal example and the proofs of the remaining theorems are given.

§1. Proof of Theorem 1

We denote the k-fold bouquet of a space X by $X_{(k)}$ and similarly for maps.

We use λ to denote a positive integer whose mod p reduction is a primitive $(p-1)$-st root of unity.

If $f: \Sigma^2 X \to S^{2n+1}$ is a co-H-map, then any p-th Hilton–Hopf invariant of $\Omega\sigma \circ f^*$ is trivial, where σ is the comultiplication on S^{2n+1}. Thus f^* factors through a complex of dimension $2n(p-1)$ and $H \circ f^*$ is inessential for dimensional reasons. We note in passing that only one suspension on X is needed for this argument. Let $\lambda I: \Sigma X \to \Sigma X$ be the λ-fold sum of the identity map. We abuse notation to let λ also denote a map of degree λ on S^{2n+1}. The Hilton–Milnor theorem and the second suspension give an isomorphism of abelian groups

$$[\Sigma X, \Omega S^{2n+1}_{(2)}] \cong \prod_{w \in J} [\Sigma X, \Omega S^{2n|w|+1}]$$

where J is a Hall basis for the free Lie algebra on two generators and $|w|$ denotes weight. Expressing $\Omega\sigma \circ f^*$ in terms of components yields

$$\Omega\sigma \circ f^* = \prod_{w} h_w(f^*), \; h_w: \Omega S^{2n+1} \to \Omega S^{2n|w|+1}.$$

Since S^{2n+1} is an H-space, $f^* \circ \lambda I \simeq \Omega\lambda \circ f^*$, thus

$$\Omega\sigma \circ f^* \circ \lambda I \simeq \Omega(\lambda \vee \lambda) \circ \Omega\sigma \circ f^*.$$

Expressing this formula in terms of components yields

$$(\lambda^{|w|} - \lambda) h_w(f^*) = 0 \qquad \text{for all } w.$$

In particular $h_w(f^*) = 0$ for $1 \leqq |w| < p$ since $\lambda^{|w|} - \lambda$ is not divisible by p. If $H \circ f^*$ is null, then f^* factors through a complex of dimension $2n(p-1)$, hence $h_w(f^*) = 0$ for $|w| \geqq p$ by dimensional arguments.

§2. The co-A-deviation

The topic of higher associativity for maps of H-spaces is well developed. For homotopy associativity, the invariants have a number of useful geometric properties. These properties are developed in [Z_2]. Dual notions make sense for co-H-spaces and we develop some of their properties here. Since the proofs follow the same format as in the case for H-spaces, we suppress details, leaving only essential ingredients.

Spaces have non-degenerate basepoints, all maps and homotopies preserve basepoints and cones, suspensions etc. are reduced.

Let $f: (\Sigma X, \tau) \to (Y, \sigma)$ be a co-H-map of homotopy associative co-H-spaces, where τ is the suspension cogroup structure. Let F be a primitive homotopy from $(f \vee f) \circ \tau \to \sigma \circ f$. Thus in addition to connecting the named maps, F has a property guaranteeing that the induced map $\bar{\sigma}$ on the mapping cone C of f is a comultiplication; the following diagram commutes up to homotopy

$$\begin{array}{ccc} C & \xrightarrow{\bar{\sigma}} & C \vee C \\ & \overset{\Delta}{\searrow} & \downarrow \\ & & C x C \end{array}$$

This point is discussed in [BH]. In particular, no loss of generality is entailed by requiring primitive homotopies for co-H-maps.

The co-A-deviation is constructed from F and homotopies associated with τ and σ. We use the following notation for stringing together a list of homotopies. Given $H_1, \ldots, H_n : Z \times I \to W$ such that $H_i(z, 1) = H_{i+1}(z, 0)$, define

$$\{H_1, \ldots, H_n\} : Z \to W^I$$

by

$$\{H_1, \ldots, H_n\}(z, t) = H_i(z, nt - i + 1) \qquad \text{if } i - 1 \leqq nt \leqq i.$$

We also write $H'(z, t)$ for $H(z, 1 - t)$.

Now define

$$A(f, F) : (\Sigma X, *) \to (\Lambda Y_{(3)}, c)$$

by stringing six homotopies in

$$A(f, F) = \{(f \vee F) \circ \tau, (1 \vee \sigma) \circ F, L_Y \circ f, (\sigma \vee 1) \circ F', (F' \vee f) \circ \tau, f_{(3)} \circ L'_X\}.$$

In this formula, Λ is the free loop functor and c is the map sending all of the circle to the basepoint of $Y_{(3)}$. We regard the circle as the unit interval with endpoints identified to the basepoint. The homotopy L_X is from $(1 \vee \tau) \circ \tau$ to $(\tau \vee 1) \circ \tau$, and likewise for L_Y, σ. The co-A-deviation is a homotopy class of maps obtained from a normalization of $A(f, F)$. Let $u : (\Sigma X, *) \to (\Lambda Y_{(3)}, c)$ be the map which for each $x \in \Sigma X$ sends every point in the circle to $f_{(3)} \circ (1 \vee \tau) \circ \tau(x)$. Since all maps are pointed, the evaluation at the basepoint of the difference $A(f, F) - u$ (with respect to the suspension coordinate in ΣX) is homotopic to the constant map. Hence there is a map

$$A_*(f, F) : (\Sigma X, *) \to (\Omega Y_{(3)}, *)$$

whose composition with the inclusion of $\Omega Y_{(3)}$ into $\Lambda Y_{(3)}$ is homotopic to $A(f, F) - u$. Since this inclusion induces a split injection of generalized homotopy groups, the homotopy class of A_* is unique. We use this notation to mean the homotopy class.

DEFINITION. f is a *co-A-map* provided there is a primitive homotopy F such that $A_*(f, F)$ is the homotopy class of the constant map.

It is easy to check that the comultiplication $\bar{\sigma}$ on C induced by F is homotopy associative provided $A_*(f, F) = *$.

We now turn to some of the properties of this construction.

(2.1) *Composition*. If $g : Z \to X$, then

$$A_*(f \circ \Sigma g, F \circ \Sigma g) = A_*(f, F) \circ \Sigma g.$$

If both (Y, σ) and (Z, ν) are suspension cogroups and $g : Y \to Z$ is a suspension, then

$$A_*(g \circ f, g_{(2)} \circ F) = \Omega g_{(3)} \circ A_*(f, F).$$

The point in having suspension requirements on g is to perform the factorization without introducing new homotopies. The suspension coordinate on ΣX allows us to write $(\Sigma g \vee \Sigma g) \circ \tau = \tau \circ \Sigma g$, and similarly for L_X.

(2.2) *Dependence on the homotopy class of f.* If $f \sim g$ via homotopy K, then $A_*(f, F) = A_*(g, G)$ where $G = \{(K \vee K)' \circ \tau, F, \sigma \circ K\}$.

These two properties are all that is needed from this section for one of the implication directions of Theorem 2.

(2.3) *Dependence on the homotopy F.* If we vary F among primitive homotopies, the variation of A_* is described in terms of a certain coboundary map,

$$\delta_2 : [\Sigma X, \Omega Y_{(2)}] \to [\Sigma X, \Omega Y_{(3)}].$$

We define δ_k in general. Let

$$E_1, E_{k+1} : Y_{(k)} \to Y_{(k+1)}$$

be the obvious embeddings opposite the first and last factors respectively. Let

$$\sigma_i : Y_{(k)} \to Y_{(k+1)}, \qquad 1 \leqq i \leqq k$$

be the map

$$\sigma_i = 1 \vee \cdots \vee \sigma \vee \cdots \vee 1$$

with σ in the i-th place. Then for $f: \Sigma X \to \Omega Y_{(k)}$, define

$$\delta_k(f) = \Omega E_1 \circ f + \sum_{i=1}^{k} (-1)^i \Omega \sigma_i \circ f + (-1)^{k+1} \Omega E_{k+1} \circ f.$$

Since the groups involved are abelian, δ is bilinear as an operator on f and $\delta^2 = 0$. We call δ_k the *geometric coboundary*. If F_1, F_2 are primitive homotopies from $(f \vee f) \circ \tau$ to $\sigma \circ f$, then

$$A_*(f, F_1) - A_*(f, F_2) = \delta_2 w$$

where

$$w = \{F_1, F_2'\} - u$$

with u constant at $(f \vee f) \circ \tau$. Since primitive homotopies (more precisely, their differences) are classified by maps from ΣX to the fibre of $\Omega Y_{(2)} \to \Omega(Y \times Y)$, modification of a primitive homotopy by such a map produces another primitive homotopy, so we write

$$A_*(f, F + w) = A_*(f, F) + \delta_2 w.$$

In particular, f is a co-A-map if and only if $A_*(f, F) = \delta_2 w$ some w.

(2.4) *Zabrodsky's formula*. The geometric fact enabling us to make the required calculations is a relation between co-H- and co-A-deviations discovered by Zabrodsky (in the dual case) in his study of homotopy associativity of H-spaces [Z_1]. Consider the situation

$$(\Sigma Z, \nu) \xrightarrow{\Sigma g} (\Sigma X, \tau) \xrightarrow{f} (Y, \sigma)$$

where $f \circ \Sigma g$ is null-homotopic. A given null-homotopy determines a map from the mapping cone C of Σg

$$h : C \to Y.$$

The map h need not be a co-H-map. Its co-H-deviation, determined by $\sigma \circ h$, is a map from $\Sigma^2 Z$ to $Y_{(2)}$ and is represented by the adjoint of

$$D_h = \{(h \vee h) \circ v, F \circ g, \sigma \circ h'\} : \Sigma Z \to \Omega Y_{(2)}$$

where h is regarded as a homotopy from $*$ to $f \circ \Sigma g$. Here the suspension coordinate is used to write $\Sigma g_{(2)} \circ v = \tau \circ \Sigma g$ avoiding an extra homotopy in our formulas. Then Zabrodsky's formula reads

$$\delta_2 D_h = A_*(f \circ \Sigma g, F \circ \Sigma g).$$

Of course A_* lies in the image of δ_2. The usefulness of this formulation is its precision. It will enable us to analyze the co-A-deviation for maps where the mapping cone provides no information.

(2.5) *Compatibility with self-maps.* We impose more suspension coordinates and work with $f: (\Sigma^2 X, \tau) \to (\Sigma^2 Y, \sigma)$ where both τ and σ are the suspension cogroup structures. Let λ_1, λ_2 be the λ-fold sums of the identity of $\Sigma^2 X$, $\Sigma^2 Y$ respectively. We use the other suspension coordinate, so that $(\lambda_1 \vee \lambda_1) \circ \tau = \tau \circ \lambda_1$ and similarly for λ_2, σ. Suppose also that there is a homotopy K from $\lambda_2 \circ f$ to $f \circ \lambda_1$. Let $D : \Sigma^2 X \to \Omega(\Sigma^2 Y_{(2)})$ be given by

$$D = \{(\lambda_2 \vee \lambda_2) \circ F, \sigma \circ K, F' \circ \lambda_1, (K \vee K)' \circ \tau\} - u$$

where $u = (\lambda_2 \vee \lambda_2) \circ (f \vee f) \circ \tau$. Then we have

$$\Omega(\lambda_2 \vee \lambda_2 \vee \lambda_2) \circ A_*(f, F) - A_*(f, F) \circ \lambda_1 = \delta_2 D.$$

This completes our discussion of geometric properties of the co-A-deviation. Of course analogous results can be established in the standard homotopy category using the co-operation of mapping cones. But this approach makes the application of the Hilton–Milnor theorem cumbersome.

Next we turn to some algebra used to analyze the geometric coboundary in terms of the Hilton–Milnor theorem. Let $\iota_1, \ldots, \iota_k \in \Pi_{2n}(\Omega S^{2n+1}_{(k)})$ be the fundamental classes. Then the map δ_k is given as follows. Let $\partial_i : S^{2n}_{(k)} \to \Omega S^{2n+1}_{(k+1)}$, $0 \leqq i \leqq k+1$ be given by

$$\partial_i(\iota_j) = \begin{cases} \iota_j & j \leqq i-1 \\ \iota_j + \iota_{j+1} & j = i \\ \iota_{j+1} & j \geqq i+1 \end{cases}$$

and let $\partial_i^\wedge : \Omega S^{2n+1}_{(k)} \to \Omega S^{2n+1}_{(k+1)}$ be the canonical extension. Then

$$\delta_k(f) = \sum_{i=0}^{k+1} (-1)^i \partial_i^\wedge \circ f = \left(\sum_{i=0}^{k+1} (-1)^i \partial_i^\wedge \right) \circ f$$

where addition is with respect to the loop structure, which distributes as indicated. ($\Sigma(-1)^i \partial_i^\wedge$ is not the same as $(\Sigma(-1)^i \partial_i)^\wedge$.) Let $L_k \subset \Pi_* \Omega S^{2n+1}_{(k)}$ be the free Lie algebra generated by the fundamental classes. The *Lazard differential* [L] is given by

$$d_k : L_k \to L_{k+1}, \qquad d_k = \sum_{i=0}^{k+1} (-1)^i d_i$$

where d_i is defined by the same formula as ∂_i. There is a grading by weight on L_k which is respected by d_k. The complex (L_k, d_k) is called a *Lie analyzer* and its cohomology groups are written as

$$H^k_m = \ker d_k : L_{k,m} \to L_{k+1,m} / im\, d_{k-1} : L_{k-1,m} \to L_{k,m},$$

where $L_{k,m}$ is the subspace of L_k spanned by terms of weight m. A different, but chain equivalent differential is introduced by Barratt in [B$_1$], [B$_2$]. In Barratt's notation, H^k_m is written $H_{k,m}$. These groups are very difficult to determine. In this paper, we use the information that $H^2_p = 0$ and $H^3_p = Z/_pZ$. We could avoid using the second of these facts were a certain geometric proposition known. This appears at the appropriate place in the paper.

On occasion, our arguments involve inferences about the Lazard coboundary from information about the geometric coboundary. This step occurs in the assertions about p-divisibility of certain maps. Of course, the general relation between the two is expressed in terms of the distributive law. For our purposes, it is enough to keep track of weights. We turn to this machinery now.

Let J^k be a Hall basis for L_k. Using the Hilton–Milnor theorem, we filter $[\Sigma^2 X, \Omega S^{2n+1}_{(k)}]$ by weight;

$$F_m = \prod_{l \geqq m} \prod_{w \in J^k_l} [\Sigma^2 X, \Omega S^{2nl+1}].$$

The basis element w of weight l can be regarded as a Samelson product $w: S^{2nl} \to \Omega S^{2n+1}_{(k)}$ with extension $w^\wedge : \Omega S^{2nl+1} \to \Omega S^{2n+1}_{(k)}$. In terms of Samelson products, the distributive law takes the following form. Given $\beta_1, \ldots, \beta_r : S^d \to \Omega Z$, $\alpha : T \to \Omega S^{d+1}$ with T a finite complex, then

$$(\beta_1 + \cdots + \beta_r)^\wedge \circ \alpha = \sum_{i=1}^{r} \beta_i^\wedge \circ \alpha + \sum_{w \in J^r} w^\wedge(\beta_1, \ldots, \beta_r) \circ h_w(\alpha)$$

where $h_w : \Omega S^{d+1} \to \Omega S^{d|w|+1}$ and by definition

$$w^\wedge(\beta_1, \ldots, \beta_r) = (\beta_1, \ldots, \beta_r)^\wedge \circ w^\wedge$$

where $(\beta_1, \ldots, \beta_r) : S^d_{(r)} \to \Omega Z$ is β_i in the i-th factor.

The map $w \otimes f \to w^\wedge \circ f$ induces an isomorphism

$$L_{k,m} \otimes [\Sigma^2 X, \Omega S^{2nm+1}] \to F_m / F_{m+1}$$

and $L_{k,m} = \operatorname{span} J^k_m$. Furthermore, on the associated graded object, the geometric and Lazard coboundaries agree,

$$E^0_m(\delta_k) = d_k \otimes 1.$$

We write $H^k_m(C)$ for the cohomology of the complex $(L_k \otimes C, d_k \otimes 1)$. In the spectral sequence associated with the filtration by weight, we have

$$E^1_{m,k} = H^k_m([\Sigma^2 X, \Omega S^{2nm+1}])$$

and

$$d^r : E^r_{m,k} \to E^r_{m+r,k+1} \qquad \text{for } r \geqq 0.$$

Now if X is a finite complex, the distributive law yields that if $w \in L_{k,m}$, then

$$\delta_k(w \otimes f) = \begin{Bmatrix} d_k w \otimes f & \text{in } F_m/F_{m+1} \\ 0 & \text{in } F_l/F_{l+1} \text{ unless } l \equiv 0 \bmod m \end{Bmatrix}.$$

So on E^r_{m*}, d^r is 0 unless $r \equiv 0 \bmod m$. We say that $b \in [\Sigma^2 X, \Omega S^{2n+1}_{(k)}]$ is *homogeneous of degree m* provided all its projections to F_l/F_{l+1} are 0 if $l \neq m$. Then we have

Lemma 2.5. *If X is a finite complex, b is a homogeneous element of prime degree p and $b = \delta c$ is a geometric coboundary, then $[b]$ in $L_{k,p} \otimes [\Sigma^2 X, \Omega S^{2np+1}]$ is a Lazard coboundary.*

§3. The universal example

Consider the map $\Omega i : \Omega J_{p-1} S^{2n} \to \Omega^2 S^{2n+1}$ from Toda's fibration. Let $K = S^{2n-1} \cup e^{2np-2}$ where the attaching map, denoted w_n, generates the kernel of the double suspension. Let $j : K \to \Omega J_{p-1} S^2 n$ denote the inclusion of the bottom two cells. A retraction $r : \Sigma\Omega J_{p-1} S^{2n} \to \Sigma K$, constructed by Moore, is described in [S$_2$]. The following is proved in [H].

LEMMA 3.1. *Let $h : \Omega J_{p-1} S^{2n} \to \Omega Y$ be an H-map. Then the adjoint h^* satisfies $h^* \simeq (h \circ j)^* \circ r$.*

Taking $h = \Omega i$ in the lemma and then adjointing again yields

$$\Omega i^{**} \simeq g \circ \Sigma r$$

where $g : \Sigma^2 K \to S^{2n+1}$ is $(\Omega i \circ j)^{**}$. Theorem 1 yields that g is a co-H-map. It is our universal example. We use G to denote a primitive homotopy for g.

REMARK. $\Sigma^2 K \simeq S^{2n+1} \vee S^{2np}$. If $n > 1$, g is homotopic to the sum of the identity map and suspension. There are no maps from ΣK to S^{2n} extending the identity, so this representation of g cannot desuspend. If $n = 1$, then $K \simeq S^1 \vee S^{2p-2}$ and $g = 1 + \alpha_1$, where α_1 generates $\Pi_{2p} S^3$.

The next step is to tie Toda's map into this setting. We have chosen T to be an H-map [G$_1$], so (3.1) yields that $T^* \simeq (T \circ j)^* \circ r \simeq \Sigma v \circ r$ where $v : K \to S^{2np-2}$ is the pinch map. For dimensional reasons, the co-A-deviation

$$A_*(g, G) : \Sigma^2 K \to \Omega S^{2n+1}_{(3)}$$

factors as $A_G \circ \Sigma^2 v$ for a unique homotopy class

$$A_G \in \Pi_{2np}(\Omega S^{2n+1}_{(3)}).$$

We write its canonical extension

$$A_{\hat{G}} : \Omega S^{2np+1} \to \Omega S^{2n+1}_{(3)}.$$

Using (2.1) we have

$$\begin{aligned} A_*(g \circ \Sigma r, G \circ \Sigma r) &= A_G \circ \Sigma T^* \\ &= A_{\hat{G}} \circ (\Sigma^2 T^*)^* \end{aligned}$$

and $(\Sigma^2 T^*)^* : \Sigma^2 \Omega J_{p-1} S^{2n} \to \Omega S^{2np+1}$ is double adjoint to $\Omega E^2 \circ T$. (We are using $*$ to denote the adjoint operator, with the direction determined by context.) Now suppose $f:\Sigma^2 X \to S^{2n+1}$ is a co-H-map. Apply Theorem 1 to write

$$f \simeq g \circ \Sigma r \circ \Sigma^2 \bar{f}.$$

Then by (2.2), there is a primitive homotopy for f induced from G, and we can write the *factorization formula*

$$A_*(f, F) = A_{\hat{G}} \circ (\Omega E^2 \circ T \circ \bar{f})^{**}.$$

We now prove part of Theorem 2; $S \circ f^{**} \sim *$ implies f is a co-A-map. Under this hypothesis, we have

$$\Omega E^2 \circ T \circ \bar{f} \simeq \Omega^3 p \circ h,\ h : X \to \Omega^3 S^{2np+1}.$$

Changing $\bar{f}$ to $\bar{f} - \Omega\partial \circ h$, with the corresponding change in F, using the facts that T is an H-map and

$$\Omega E^2 \circ T \circ \Omega\partial \simeq \Omega^3 p,$$

the factorization formula yields that $A_*(f, F) = *$ for the new F.

Our next task is to get sufficient information regarding the non-triviality of A_G. By itself, g seems impossible to analyze. Certain hints can be gleened from [B] and [BC]. From these papers one sees a connection between g and w_{n+1}. In his study of desuspension [G_2], Gray supplies the information that we use to go forward.

In [G_2], spaces Y_n and Y_n', $n \geqq 1$, are constructed with the following properties, $q = 2p - 2$:

(i) $Y_n \simeq S^{2n+1} \cup e^{2n+1+q} \cup \cdots \cup e^{2np+1}$, there are a total of $(n+1)$ cells, one in each dimension of the form $2n+1+iq$, $0 \leqq i \leqq n$. Furthermore, $\mathscr{P}^n$ is non-zero on $H^{2n+1}(Y_n; Z/_pZ)$.

(ii) There is a map $g_n : \Sigma^{2p-1} Y_n' \to S^{2n+1}$ with Y_n equal to the cofibre of g_n. In case $n = 1$, $g_1 : S^{2p} \to S^3$ is α_1. In case $n = 2$, $g_2 : S^{2p+2} \cup e^{4p} \to S^5$ is part of the Toda bracket $\langle \alpha_1, \alpha_1, \alpha_1 \rangle$.

(iii) g_n factors as a composition

$$\Sigma^{2p-1} Y_n' \xrightarrow{\Sigma^2 a} \Sigma^2 K \xrightarrow{\Sigma^2 j} \Sigma^2 \Omega J_{p-1} S^{2n} \xrightarrow{\Omega i^{**}} S^{2n+1}$$

thus $g_n \simeq g \circ \Sigma^2 a$.

We prove that g is not a co-A-map, by proving

Proposition 3.2. *g_n is not a co-A-map.*

Proof. For dimensional reasons, g_n has finite order in $[\Sigma^{2p-1}Y'_n, S^{2n+1}]$, so there is a null composition

$$S^{2np} \xrightarrow{t} \Sigma^{2p-1}Y'_n \xrightarrow{g_n} S^{2n+1}$$

such that t followed by the pinch map to S^{2np} has degree some power of p, say p^b. We also take t to be a suspension. Let C_t be the mapping cone of t and

$$h : C_t \to S^{2n+1}$$

extend g_n. Using property (i), we obtain the information that $\beta_b \mathcal{P}^n$ is non-zero on $H^{2n+1}(C_h, Z/_pZ)$, where C_h is the cofibre of h, and β_b is the b-th order Bockstein. Hence C_h cannot be a retract of $\Sigma\Omega C_h$, and is therefore not a co-H-space. This means that no matter what homotopies are used, the co-H-deviation of h

$$D_h : S^{2np} \to \Omega S^{2n+1}_{(2)}$$

is not zero. Furthermore, use of self-maps, as in the proof of Theorem 1, shows that $D_h \in L_{2,p}$. Applying Zabrodsky's formula (2.4) and (2.1) yields

$$\delta_2 D_h = p^b A_*(g_n, G_n)$$

for any primitive homotopy G_n. But the facts $L_{1,p} = 0$ and $H^2_p = 0$ imply that $\delta_2 = d_2 : L_{2,p} \to L_{3,p}$ is monic. Hence $A_*(g_n, G_n) \neq 0$.

Remark. If $(p-1)$ does not divide n, then Berstein's argument [B] shows that Y_n is not a cogroup. Theorem 1 could have been used to show h is not a co-H-map, but the argument is longer than that given.

Next, we use the information from (3.2) to obtain information about $A_G \in \Pi_{2np}(\Omega S^{2n+1}_{(3)})$.

Proposition 3.3. *There is a primitive homotopy G for g such that $A_G \in L_{3,p}$ is a cocycle and generates $H^3_p = Z/_pZ$.*

Proof. Let J be a Hall basis for L_3. Using the Hilton–Milnor theorem, we identity A_G with

$$\prod_{w \in J} h_w(A_G),\ h_w(A_G) \in \Pi_{2np}(\Omega S^{2n|w|+1}).$$

Collect terms by weight, and write

$$A_G^l = \prod_{|w|=l} h_w(A_G).$$

Now $l \leqq p$ for dimensional reasons. Expressing the formula from (2.5) in terms of components yields

$$(\lambda^l - \lambda)A_G^l = \delta_2 D^l.$$

Using (2.3) we can change the homotopy by

$$G' = G - \sum_{l<p} (\lambda^l - \lambda)^{-1} D^l$$

to obtain an element which is homogeneous of degree p. Let $A_G \in L_{3,p}$ be such an element. In the proof of (3.2) we obtained the information that $p^b A_G$ is a coboundary. Since $L_{4,p}$ is torsion free, it follows that A_G is a cocycle. From (3.2) we know that A_G is not a coboundary, so (3.3) follows.

REMARK. Let $i: S^{2n+1} \to P^{2n+2}(p)$ be the inclusion of the bottom cell. If one knows that $i \circ g_n$ is not a co-A-map, then a weaker, but just as useful form of (3.3) can be proved; A_G generates a cyclic summand of order p in H_p^3. This argument does not rely on Barratt's calculations. For values of n not divisible by $(p-1)$, this fact can be established using Berstein's approach [B].

We now prove the other half of Theorem 2; if $f: \Sigma^2 X \to S^{2n+1}$ is a co-A-map, then $S \circ f^{**} \sim *$. First assume X is a finite complex. Since f is assumed to be a co-A-map, $A_*(f, F) = \delta_2 w$ holds. Using the information in (3.3) and applying (2.5) yields that

$$b = A_G \otimes (\Omega E^2 \circ T \circ \bar{f})^{**}$$

is a Lazard coboundary in $L_{3,p} \otimes [\Sigma^2 X, \Omega S^{2np+1}]$. Since A_G generates $H_p^3 = Z/_pZ$, the universal coefficient theorem applied to $H_p^3([\Sigma^2 X, \Omega S^{2np+1}])$ yields that $(\Omega E^2 \circ T \circ \bar{f})^{**}$ is divisible by p. Using the H-structure on S^{2np+1}, we can write

$$\Omega E^2 \circ T \circ \bar{f} = \Omega^3 p \circ h$$

for some $h: X \to \Omega^3 S^{2np+1}$. For general X, the above argument shows that $S \circ f^{**}$ is a phantom map. But, by [N], $\Omega^2 S^{2np+1}\{p\}$ has H-space exponent p and each homotopy group is finite, so phantom maps to this space are trivial.

Next we prove Theorem 3. Suppose $f: \Sigma^2 X \to S^3$ is a co-A-map. Consider the following diagram

$$
\begin{array}{ccccccc}
 & & \Omega^3 S^{2p+1} & \searrow & & & \\
 & & \downarrow & & & & \\
X & \xrightarrow{\bar{f}} & \Omega J_{p-1} S^2 & \xrightarrow{T} & \Omega S^{2p-1} & \xrightarrow{\Omega E^2} & \Omega^3 S^{2p+1} \\
 & & & & \downarrow{\scriptstyle \Omega\alpha_1^*} & & \downarrow \\
 & & & & \Omega^2 S^3\langle 3\rangle & \underset{\Omega^2\gamma}{\longleftarrow} & \Omega^2 S^{2p+1}\{p\}
\end{array}
$$

where $\gamma : S^{2p+1}\{p\} \to S^3\langle 3\rangle$ exists because $\Sigma\alpha_1 : S^{2p+1} \to BS^3\langle 3\rangle$ has order p. This construction gives the commutative square on the right.

By Theorem 2,

$$\Omega\alpha_1^* \circ T \circ \bar{f} \sim *.$$

So $T \circ \bar{f}$ factors through $\Omega^3 S^{2p+1}$. Thus $\bar{f}$ can be altered to a map factoring through S^1, the fibre of T. The clause about $\Pi_* S^3$ follows because the image of the suspension map from $\Pi_* S^2$ is 2-primary.

Proof of Theorem 4. Suppose $f : X \to S^{2p-1}$ satisfies $\alpha_1 \circ \Sigma f \sim *$. We have

$$A_*(\alpha_1 \circ \Sigma f, G \circ \Sigma f) = A_*(\alpha_1, G) \circ \Sigma f$$

and divisibility by p of $\Sigma^2 f$ follows as in the proof of Theorem 2. The converse is immediate, because if $\Sigma^2 f$ is divisible by p, then the suspension of $\alpha_1 \circ \Sigma f$ is null, but suspension is monomorphism.

References

[B] I. Berstein, *A note on spaces with non-associative comultiplication*, Proc. Camb. Phil. Soc. **60** (1964), 353–354.

[BH] I. Berstein and J. R. Harper, *Cogroups which are not suspensions*, to appear.

[B$_1$] M. G. Barratt, *A theorem on the homotopy of a certain differential group*, Quart. J. Math. **11** (1960), 275–286.

[B$_2$] M. G. Barratt, *Ring stacks*, J. London Math. Soc. **36** (1961), 480–495.

[BC] M. G. Barratt and P. H. Chan, *A note on a conjecture of Ganea*, J. London Math. Soc. **20** (1979), 544–548.

[CMN] F. R. Cohen, J. C. Moore and J. A. Neisendorfer, *The double suspension and exponents of the homotopy groups of spheres*, Ann. of Math. **110** (1979), 549–565.

[G$_1$] B. Gray, *On Toda's fibrations*, Math. Proc. Camb. Phil. Soc. **97** (1985), 289–298.

[G$_2$] B. Gray, *Desuspension at an odd prime*, in *Algebraic Topology Aarhus 1982*, Springer Lecture Notes in Math. **1051**, Springer-Verlag, Berlin, 1984, pp. 360–370.

[H] J. R. Harper, *A proof of Gray's conjecture*, to appear.

[L] M. Lazard, *Lois de groupes et analyseurs*, Ann. Sci. Ec. Norm. Super. **52** (1955), 299–400.

[N] J. A. Neisendorfer, *Properties of certain H-spaces*, Quart. J. Math. **34** (1983), 201–209.

[S$_1$] P. Selick, *Odd primary torsion in* $\Pi_k S^3$, Topology **17** (1978), 407–412.

[S_2] P. Selick, *A spectral sequence concerning the double suspension*, Invent. Math. **64** (1981), 15–24.

[T] H. Toda, *On the double suspension* E^2, J. Inst. Polytech. Osaka City Univ. Ser. **A7** (1956), 103–145.

[Z_1] A. Zabrodsky, *Implications in the cohomology of H-spaces*, Ill. J. Math. **14** (1970), 363–375.

[Z_2] A. Zabrodsky, *Hopf Spaces*, North-Holland, Amsterdam, 1976.